Gaussian Process based Model Predictive Control for Linear Time Varying Systems

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Outline

Abbreviations

- **GP: Gaussian process**
- **MPC: Model predictive control**
- **LTV: Linear time varying**
- **CG: Conjugate gradient**

Problem Formulation

Linear time-varying system:

$$\mathbf{x}_{k+1} = \mathbf{A}_k \mathbf{x}_k + \mathbf{B}_k \mathbf{u}_k + \mathbf{w}_k \tag{1}$$

- $\mathbf{x}_k \in \mathbb{R}^n$, $\mathbf{u}_k \in \mathbb{R}^m$, $\mathbf{w}_k \in \mathbb{R}^n$;
- State matrix $\mathbf{A}_k$ and Input matrix $\mathbf{B}_k$ are unknown;

Unconstrained MPC trajectory tracking problem:

$$\min \mathbf{V}_k^* = \min_{\mathbf{U}_k^*} \sum_{i=1}^{H} \left\{ \|\mathbf{x}_{k+i} - \mathbf{r}_{k+i}\|_{\mathbf{Q}}^2 + \|\mathbf{u}_{k+i-1}\|_{\mathbf{R}}^2 \right\} \tag{2}$$

- $\mathbf{U}_k^* = [\mathbf{u}_{k-1}^*, \cdots, \mathbf{u}_{k+H-1}^*]^T$;

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Problems:

- Identification of unknown LTV system
- Solving the MPC problem effectively

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Contributions:

- GP based MPC;
- Gradient based solution;

GP Learning of Unknown LTV

Gaussian Process Models:

GP Learning of Unknown LTV

Gaussian Process Models:

- Probabilistic data-driven model;

GP Learning of Unknown LTV

Gaussian Process Models:

- Probabilistic data-driven model;
- Specifying a squared exponential covariance function $\mathbf{K}(\mathbf{x}_{k,i}, \mathbf{x}_{k,j}, \boldsymbol{\theta})$;
  $\mathbf{K}(\tilde{\mathbf{x}}_i, \tilde{\mathbf{x}}_j) = \sigma_s^2 \exp(-\frac{1}{2}(\tilde{\mathbf{x}}_i - \tilde{\mathbf{x}}_j)^T \boldsymbol{\Lambda}(\tilde{\mathbf{x}}_i - \tilde{\mathbf{x}}_j)) + \sigma_n^2$

GP Learning of Unknown LTV

Gaussian Process Models:

- Probabilistic data-driven model;
- Specifying a squared exponential covariance function $\mathbf{K}(\mathbf{x}_{k,i}, \mathbf{x}_{k,j}, \boldsymbol{\theta})$;
- $\mathbf{x}_k^* \sim \mathcal{N}\{\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k\}$;

$$\boldsymbol{\mu}_k = \mathbf{K}(\tilde{\mathbf{x}}_k^*, \tilde{\mathbf{X}})(\mathbf{K}(\tilde{\mathbf{X}}, \tilde{\mathbf{X}}) + \sigma_n \mathbf{I})^{-1} \mathbf{X}_k \tag{3a}$$

$$\boldsymbol{\Sigma}_k = \mathbf{K}(\tilde{\mathbf{x}}_k^*, \tilde{\mathbf{x}}_k^*) - \mathbf{K}(\tilde{\mathbf{x}}_k^*, \tilde{\mathbf{X}})(\mathbf{K}(\tilde{\mathbf{X}}, \tilde{\mathbf{X}}) + \sigma_n \mathbf{I})^{-1} \mathbf{K}(\tilde{\mathbf{X}}, \tilde{\mathbf{x}}_k^*) \tag{3b}$$

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- Evaluation of model uncertainty $\rightarrow \boldsymbol{\Sigma}_k$;

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Unknown LTV Learning:

- GP Inputs: $\tilde{\mathbf{x}}_k = (\mathbf{x}_k, \mathbf{u}_k)$;
- GP Outputs: $\delta\mathbf{x}_{k+1} = \mathbf{x}_{k+1} - \mathbf{x}_k$;
- Hyperparameters learning: Minimize the negative log of likelihood (CG);

Gaussian Process based Model Predictive Control for Linear Time Varying Systems

GP based MPC

Problem Reformulation:

$$\begin{aligned}\mathbb{E}\Big[\mathbf{V}_k^*\Big] &= \underbrace{\mathbb{E}\Big[\sum_{i=1}^{H}\big\{\,\|\mathbf{x}_{k+i}-\mathbf{r}_{k+i}\|_{\mathbf{Q}}^2+\|\mathbf{u}_{k+i-1}\|_{\mathbf{R}}^2\,\big\}\Big]}_{\text{Stochastic MPC}} \\ &= \underbrace{\sum_{i=1}^{H}\Big\{\,\|\boldsymbol{\mu}_{k+i}-\mathbf{r}_{k+i}\|_{\mathbf{Q}}^2+\|\mathbf{u}_{k+i-1}\|_{\mathbf{R}}^2+\mathrm{trace}\big(\mathbf{Q}\boldsymbol{\Sigma}_{k+i}\big)\Big\}}_{\text{Deterministic MPC}}\end{aligned} \tag{4}$$

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Implementation Issues:

- Uncertainty propagation → GP predictions at uncertain inputs;

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Implementation Issues:

- Uncertainty propagation → GP predictions at uncertain inputs;
- Effective Solutions → Gradient based optimization;

Gradient based optimization

Unconstrained optimization question:

$$\mathbf{U}_k^* = \arg\min_{\mathbf{U}_k^*} \mathbb{E}\left\{\mathbf{V}_k^*\right\} \tag{5a}$$

$$\text{s.t. } \boldsymbol{\mu}_{k+1} = \mathcal{G}(\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k, \mathbf{u}_k); \tag{5b}$$

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- Gradient based linear search;

$$\mathbf{U}_{k+1} = \mathbf{U}_k + \boldsymbol{\alpha} \cdot \underbrace{\frac{\partial \mathbb{E}\{\mathbf{V}_k\}}{\partial \mathbf{U}_k}}_{\text{Gradient}}; \tag{6}$$

$$\frac{\partial \mathbb{E}\{\mathbf{V}_k\}}{\partial \mathbf{U}_k} = \frac{\partial \mathbb{E}\{\mathbf{V}_k\}}{\partial \boldsymbol{\mu}_{k+1}} \frac{\partial \boldsymbol{\mu}_{k+1}}{\partial \mathbf{U}_k} + \frac{\partial \mathbb{E}\{\mathbf{V}_k\}}{\partial \boldsymbol{\Sigma}_{k+1}} \frac{\partial \boldsymbol{\Sigma}_{k+1}}{\partial \mathbf{U}_k} + \frac{\partial \mathbb{E}\{\mathbf{V}_k\}}{\partial \mathbf{U}_k} \tag{7}$$

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- End when $\mathbb{E}\{\mathbf{V}_k\}_{\mathbf{U}_k} \leq \epsilon$;
- Multi-Start with different initial values;

Numerical Simulation

2-Inputs-2-Outputs Numerical Example:

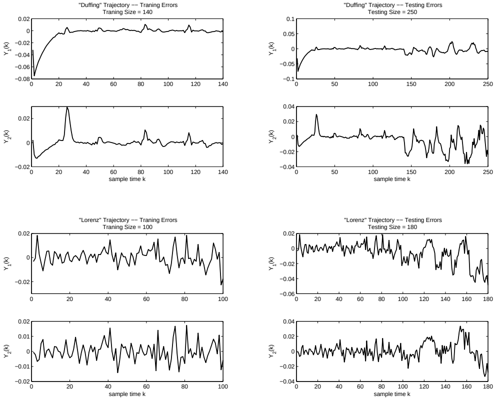
$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} a(t) & 1 \\ b(t) & 0 \end{bmatrix} \mathbf{u} + \mathbf{w} \qquad (8)$$

- $a(t) = 1 + \sin(2\pi t/1500)$ and $b(t) = \cos(2\pi t/1500)$;
- $\mathbf{w} \sim \mathcal{N}(0, 0.01)$;
- 2 trajectories: "Duffing" and "Lorenz";
- 50 repeats;
- Observations generated by using a linear MPC;
- Compare to the Nelder-Mead method (derivative-free);

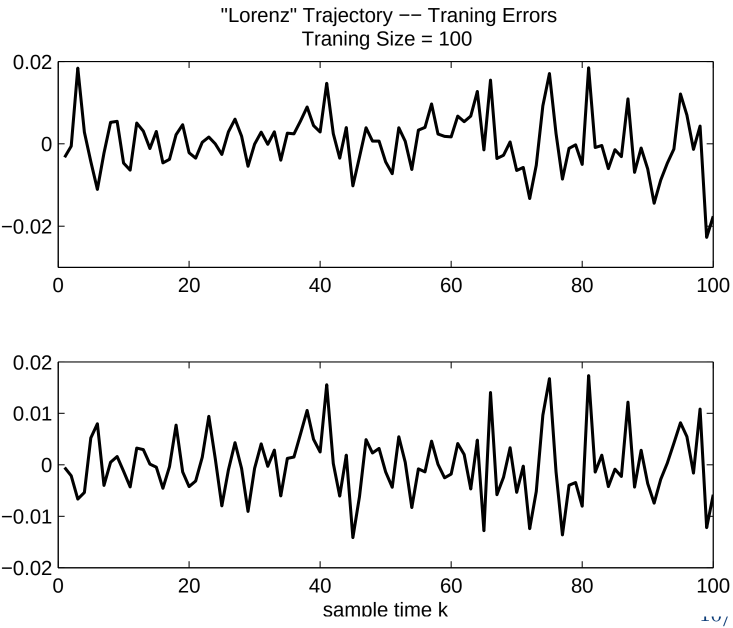
Numerical Simulation

Modelling Results:

	Training MSE	Testing MSE
“Duffing”	2.2338×10^{-4}	3.4091×10^{-4}
“Lorenz”	8.7979×10^{-5}	4.0674×10^{-4}



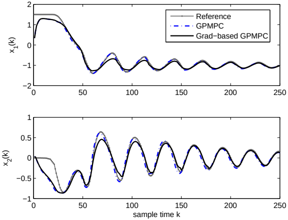
Numerical Simulation



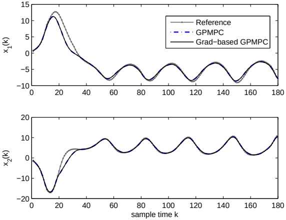
Numerical Simulation

Control Results:

	Approximately Required Time(s)	
	Grad-GPMPC	GPMPC
"Duffing"	30	70
"Lorenz"	16	37

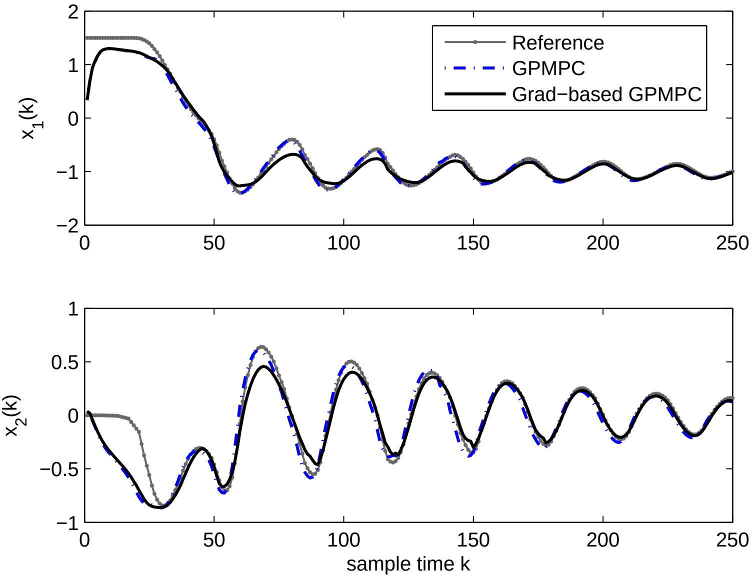


(i) "Duffing" Trajectory



(j) "Lorfenz" Trajectory

Numerical Simulation



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Future Works:

- Constrained MPC for Nonlinear unknown systems;
- Guaranteed stability and robustness against uncertainties;

Thanks!

Questions?